

Weak Solutions for the Navier-Stokes Equations with $B_{\infty\infty}^{-1(\ln)} + B_{X_r}^{-1+r, \frac{2}{1-r}} + L^2$ Initial Data*

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Abstract

In 1934 Leray proved that the Navier-Stokes equations have global weak solutions for initial data in $L^2(\mathbb{R}^N)$. In 1990 Calderón extended this result to the initial value spaces $L^p(\mathbb{R}^N)$ ($2 \leq p < \infty$). In the book “*Recent developments in the Navier-Stokes problems*” (2002), Lemarié-Rieusset extended this result of Calderón to the space $B_{\tilde{X}_r}^{-1+r, \frac{2}{1-r}}(\mathbb{R}^N) + L^2(\mathbb{R}^N)$ ($0 < r < 1$), where X_r is the space of functions whose pointwise products with H^r functions belong to L^2 , \tilde{X}_r denotes the closure of $C_0^\infty(\mathbb{R}^N)$ in X_r , and $B_{\tilde{X}_r}^{-1+r, \frac{2}{1-r}}(\mathbb{R}^N)$ is the Besov space over \tilde{X}_r . In this paper we further extend this result of Lemarié-Rieusset to the larger initial value space $\dot{B}_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N) + \dot{B}_{\tilde{X}_r}^{-1+r, \frac{2}{1-r}}(\mathbb{R}^N) + L^2(\mathbb{R}^N)$ ($0 < r < 1$).

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1 Introduction

In this paper we study existence of solutions for the initial value problem of the Navier-Stokes equations:

$$\begin{cases} \partial_t \mathbf{u} - \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla P = 0 & \text{in } \mathbb{R}^N \times \mathbb{R}_+, \\ \nabla \cdot \mathbf{u} = 0 & \text{in } \mathbb{R}^N \times \mathbb{R}_+, \\ \mathbf{u}(x, t) = \mathbf{u}_0(x) & \text{for } x \in \mathbb{R}^N. \end{cases} \quad (1.1)$$

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Here $\mathbf{u} = \mathbf{u}(x, t) = (u_1(x, t), u_2(x, t), \dots, u_N(x, t))$ is an unknown N -vector function in (x, t) variables, $x \in \mathbb{R}^N$ ($N \geq 2$), $t \geq 0$, $P = P(x, t)$ is an unknown scalar function, $\mathbf{u}_0 = \mathbf{u}_0(x)$ is a given N -vector function, Δ is the Laplacian in the x variables, $\nabla = (\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_N})$, and $\mathbb{R}_+ = (0, \infty)$ (later we shall also denote $\overline{\mathbb{R}}_+ = [0, \infty)$).

Existence of solutions of the above problem is a fundamental topic in mathematical theory of the Navier-Stokes equations. The first important result on this topic was obtained by Leray in 1934 in the reference [19], where he introduced the concept of *weak solutions* and proved that the problem (1.1) has a global weak solution in the class

$$C_w([0, \infty), L^2(\mathbb{R}^N)) \cap L^\infty([0, \infty), L^2(\mathbb{R}^N)) \cap L^2([0, \infty), \dot{H}^1(\mathbb{R}^N))$$

for any initial data $\mathbf{u}_0 \in L^2(\mathbb{R}^N)$, where $C_w([0, \infty), L^2(\mathbb{R}^N))$ denotes the set of maps from $[0, \infty)$ to $L^2(\mathbb{R}^N)$ which are continuous with respect to the weak topology of $L^2(\mathbb{R}^N)$. Here and throughout this paper, for simplicity of notations we use the same notation to denote a scalar function space and its corresponding N -vector counterpart; for instance, the notation $L^2(\mathbb{R}^N)$ denotes both the space of scalar L^2 functions and the space of N -vector L^2 functions. Since $\mathbf{u} \in L^\infty([0, \infty), L^2(\mathbb{R}^N))$, i.e., $\sup_{t \geq 0} \|\mathbf{u}(t)\|_2 < \infty$, Leray's weak solutions are usually referred to as *finite energy* weak solutions in the literature. To obtain this result Leray used a smooth approximation approach based on weak compactness of bounded sets in separable Banach spaces and dual of Banach spaces. A different approach based on Picard's iteration scheme (or Banach's fixed point theorem) was introduced by Kato and Fujita in 1964 in [8], where they established local existence of mild solutions for $\mathbf{u}_0 \in H^s(\mathbb{R}^N)$ for $s \geq \frac{N}{2} - 1$, and global existence of such solutions for small initial data in $H^{\frac{N}{2}-1}(\mathbb{R}^N)$. This approach was later extended to various other function spaces, such the Lebesgue space $L^p(\mathbb{R}^N)$ for $p \geq N$ by Weissler [23], Kato [14], Fabes, Johns and Riviere [7] and Giga [10], the critical and subcritical Sobolev spaces and Besov spaces of either positive or negative orders by Kato and Ponce [15], Planchon [20], Terraneo [22], Cannon [4] and et al, the Lorentz spaces $L^{p,q}$ by Barraza [1], the Morrey-Campanato spaces $M^{p,q}$ by Giga and Miyakawa [12], the space BMO^{-1} of derivatives of BMO functions by Koch and Tataru [16], and some general Sobolev and Besov spaces over shift-invariant Banach spaces of distributions (see Definition 1.3 below) that can be continuously embedded into the Besov space $B_{\infty\infty}^{-1}(\mathbb{R}^N)$ by Lemarié-Rieusset in his expository book [18]. The literatures listed here are far from being complete; we refer the reader to see [5] and [18] for expositions and references cited therein. The largest initial value space for existence of solutions found by using this approach is the BMO^{-1} space introduced by Koch and Tataru in [16], which coincides with the Triebel-Lizorkin space $\dot{F}_{\infty 2}^{-1}(\mathbb{R}^N)$.

A third approach which combines the above two approaches was introduced by Calderón in 1990 in [3]. He proved global existence of weak solutions in the class $C_w([0, \infty), L^2(\mathbb{R}^N) + L^p(\mathbb{R}^N))$ for initial data $\mathbf{u}_0 \in L^p(\mathbb{R}^N)$ for any $2 \leq p < \infty$. Note that this result of Calderón fills the gap lying between that of Leray for $p = 2$ and those of Weissler et al mentioned before for $p \geq N$. In 1998, in an unpublished article [17], in the case $N = 3$ Lemarié-Rieusset extended this result of Calderón to the initial value space $E_2(\mathbb{R}^3)$, the closure of $C_0^\infty(\mathbb{R}^3)$ in the space

$L^2_{uloc}(\mathbb{R}^3)$ of uniformly locally L^2 -functions. Later in 2002, in his book [18], as a concluding result of that book, Lemarié-Rieusset further extended his result to the spaces $B_{\tilde{X}_r}^{s_r, q_r}(\mathbb{R}^N) + L^2(\mathbb{R}^N)$ and $B_{\tilde{X}_r}^{s_r, q_r}(\mathbb{R}^3) + E_2(\mathbb{R}^3)$ ($0 < r < 1$), where $s_r = -1 + r$, $q_r = \frac{2}{1-r}$, $X_r = X_r(\mathbb{R}^N)$ is the multiplier space of index r (see Definition 1.2 below), $\tilde{X}_r = \tilde{X}_r(\mathbb{R}^N)$ denotes the closure of $C_0^\infty(\mathbb{R}^N)$ in $X_r(\mathbb{R}^N)$, and $B_{\tilde{X}_r}^{s_r, q_r}(\mathbb{R}^N)$ is the Besov space over $\tilde{X}_r(\mathbb{R}^N)$.

On one hand, since $B_{\tilde{X}_r}^{s_r, q_r}(\mathbb{R}^N) + L^2(\mathbb{R}^N) \not\subseteq bmo^{-1}(\mathbb{R}^N)$, where $bmo^{-1}(\mathbb{R}^N)$ is the inhomogeneous version of $BMO^{-1}(\mathbb{R}^N)$, the above-mentioned result of Lemarié-Rieusset gives some new solutions of the Navier-Stokes equations that cannot be obtained from either Leray [19] or Koch and Tataru [16]. On the other hand, since $r > 0$, we see that $s_r = -1 + r > -1$, i.e., the regularity index of the space $B_{\tilde{X}_r}^{s_r, q_r}(\mathbb{R}^N) + L^2(\mathbb{R}^N)$ is larger than that of $bmo^{-1}(\mathbb{R}^N) = F_{\infty 2}^{-1}(\mathbb{R}^N)$. Inspired by these observations, in this paper we further extend the above result of Lemarié-Rieusset to the initial value space $B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N) + B_{\tilde{X}_r}^{s_r, q_r}(\mathbb{R}^N) + L^2(\mathbb{R}^N)$, where $B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)$ is a logarithmically modified function space to $B_{\infty\infty}^{-1}(\mathbb{R}^N)$ lying in between $B_{\infty\infty}^{-1}(\mathbb{R}^N)$ and $B_{\infty\infty}^s(\mathbb{R}^N)$ for any $s > -1$, i.e., $B_{\infty\infty}^s(\mathbb{R}^N) \subseteq B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N) \subseteq B_{\infty\infty}^{-1}(\mathbb{R}^N)$ with continuous embedding (see Definition 1.2 below), and it coincides with the space $B_{\infty\infty}^{-1,1}(\mathbb{R}^N)$ introduced by Yoneda in [24]. Note that by the recent work of Bourgain and Pavlović [2], the initial value problem for the 3-dimensional Navier-Stokes equations is ill-posed in $B_{\infty\infty}^{-1}(\mathbb{R}^3)$ (see Yoneda [24] for some extensions). In this paper we shall prove that this problem is well-posed in $B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)$; see Theorem 2.1 in Section 2. Thus, $B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)$ is a reasonable substitution of $B_{\infty\infty}^{-1}(\mathbb{R}^N)$ in order to get well-posedness of the Navier-Stokes initial value problem.

We point out that since $B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N) \subseteq bmo^{-1}(\mathbb{R}^N)$, the result of well-posedness of the Navier-Stokes initial value problem in $B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)$ alone does not provide us with more solutions of this problem than those ensured by the result of Koch and Tataru [16]. However, since $B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N) + B_{\tilde{X}_r}^{s_r, q_r}(\mathbb{R}^N) + L^2(\mathbb{R}^N) \not\subseteq bmo^{-1}(\mathbb{R}^N)$, and this space is clearly larger than $B_{\tilde{X}_r}^{s_r, q_r}(\mathbb{R}^N) + L^2(\mathbb{R}^N)$, our main result essentially enlarges the solution class of the Navier-Stokes initial value problem from the known results. The reason that we use the smaller well-posedness space $B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)$ instead of the larger one $bmo^{-1}(\mathbb{R}^N)$ in our result is because solutions of the Navier-Stokes equations started from $bmo^{-1}(\mathbb{R}^N)$ class initial data are not sufficiently regular, so that functions in a class of the form $\mathbb{X} + L^2(\mathbb{R}^N)$, where \mathbb{X} represents a general function space containing $bmo^{-1}(\mathbb{R}^N)$, cannot be used as an initial value space for the Navier-Stokes initial value space. This will be clear from the discussions of Sections 3 and 4 of the present paper.

Before presenting the exact statement of our result, let us first make some preparations. First we note that since

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) - (\nabla \cdot \mathbf{u})\mathbf{u},$$

where \otimes denotes the tensor product between N -vectors, the equation in the first line of (1.1) can be rewritten as follows:

$$\partial_t \mathbf{u} - \Delta \mathbf{u} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) + \nabla P = 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+. \quad (1.2)$$

Definition 1.1 Let $0 < T \leq \infty$. Let $\mathbf{u}_0 \in (\mathcal{D}'(\mathbb{R}^N))^N$ and $\nabla \cdot \mathbf{u}_0 = 0$. A vector function $\mathbf{u} \in (L^2_{loc}(\mathbb{R}^N \times (0, T)))^N \cap C([0, T], (\mathcal{D}'(\mathbb{R}^N))^N)$ is said to be a weak solution of the problem (1.1) for $0 \leq t < T$, if there exists a distribution $P \in \mathcal{D}'(\mathbb{R}^N \times (0, T))$ such that the following two conditions are satisfied:

- (i) \mathbf{u} and P satisfy (1.2) and the equation $\nabla \cdot \mathbf{u} = 0$ in $\mathbb{R}^N \times (0, T)$ in distribution sense.
- (ii) $\mathbf{u}(\cdot, 0) = \mathbf{u}_0$.

Let $\mathbb{P} = I + \nabla(-\Delta)^{-1}\nabla$ be the Helmholtz-Weyl projection operator, i.e., the $N \times N$ matrix pseudo-differential operator in \mathbb{R}^N with the matrix symbol $\left(\delta_{ij} - \frac{\xi_i \xi_j}{|\xi|^2}\right)_{i,j=1}^N$, where δ_{ij} are the Kronecker symbols. By applying the operator \mathbb{P} to both sides of (1.2) we obtain, at least formally, the following equation without the pressure P :

$$\partial_t \mathbf{u} - \Delta \mathbf{u} + \mathbb{P} \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) = 0 \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+.$$

In order for this equation to make sense, we need to restrict the discussion to *uniformly locally square integrable weak solutions* (or L^2_{uloc} weak solutions in short), which are weak solutions such that for any $\phi \in C_0^\infty(\mathbb{R}^N \times (0, T))$, $\sup_{y \in \mathbb{R}^N} \int_0^T \int_{\mathbb{R}^N} |\phi(x-y, t) \mathbf{u}(x, t)|^2 dx dt < \infty$. For such solutions, the expression $\mathbb{P} \nabla \cdot (\mathbf{u} \otimes \mathbf{u})$ is meaningful. Indeed, by choosing a function $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $\varphi(\xi) = 1$ for $|\xi| \leq 1$ and $\varphi(\xi) = 0$ for $|\xi| \geq 2$, we can write

$$\mathbb{P} \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) = \varphi(D) \mathbb{P} \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) + [I - \varphi(D)] \mathbb{P} \nabla \cdot (\mathbf{u} \otimes \mathbf{u}).$$

Clearly, $[I - \varphi(D)] \mathbb{P} \nabla$ is a continuous linear map from $(\mathcal{S}'(\mathbb{R}^N))^{N \times N}$ to $(\mathcal{S}'(\mathbb{R}^N))^N$, so that the second term on the right-hand side of the above equality is meaningful. Moreover, since each component of $\varphi(D) \mathbb{P} \nabla$ is a convolution operator with a L^1 kernel, so that it maps $(\mathbf{u} \otimes \mathbf{u})(\cdot, t) \in (L^1_{uloc}(\mathbb{R}^N))^{N \times N}$ to $(L^1_{uloc}(\mathbb{R}^N))^N$ (for fixed t), it follows that the first term on the right-hand side also makes sense. Hence, the problem (1.1) reduces into the following apparently simpler problem:

$$\begin{cases} \partial_t \mathbf{u} - \Delta \mathbf{u} + \mathbb{P} \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) = 0 & \text{in } \mathbb{R}^N \times \mathbb{R}_+, \\ \mathbf{u}(x, t) = \mathbf{u}_0(x) & \text{for } x \in \mathbb{R}^N. \end{cases} \quad (1.3)$$

For equivalence of the problems (1.1) and (1.3) in the category of L^2_{uloc} weak solutions, we refer the reader to see Theorem 11.1 of [18].

Next we recall the definitions of some function spaces (see [18] for more details):

- For $0 < r < \frac{N}{2}$, we define the *multiplier space of index r* , $X_r(\mathbb{R}^N)$, as the Banach space of locally square-integrable functions on \mathbb{R}^N such that pointwise multiplication with these functions maps boundedly $H^r(\mathbb{R}^N)$ into $L^2(\mathbb{R}^N)$. The norm of $X_r(\mathbb{R}^N)$ is given by

$$\|u\|_{X_r} = \sup\{\|uv\|_2 : v \in H^r(\mathbb{R}^N), \|v\|_{H^r} \leq 1\}, \quad \forall u \in X_r(\mathbb{R}^N).$$

$\tilde{X}_r(\mathbb{R}^N)$ denotes the closure of $C_0^\infty(\mathbb{R}^N)$ in $X_r(\mathbb{R}^N)$. We note that $L^\infty(\mathbb{R}^N) \hookrightarrow X_r(\mathbb{R}^N) \hookrightarrow X_s(\mathbb{R}^N)$ for $0 < r < s < \frac{N}{2}$.

- Let E be a shift-invariant Banach space of distributions on \mathbb{R}^N . For $s \in \mathbb{R}$ and $1 \leq q \leq \infty$, we define the *Besov space over E of index (s, q)* , $B_E^{s,q}(\mathbb{R}^N)$, as the Banach space of temperate distributions u on \mathbb{R}^N such that $S_0 u \in E$, $\Delta_j u \in E$ for all $j \in \mathbb{N}$, and $\{2^{js} \|\Delta_j u\|_E\}_{j=1}^\infty \in l^q$, where S_0 and Δ_j are the frequency-localizing operator appearing in the Littlewood-Paley decomposition $u = S_0 u + \sum_{j=1}^\infty \Delta_j u$. The norm of $B_E^{s,q}(\mathbb{R}^N)$ is given by

$$\|u\|_{B_E^{s,q}} = \begin{cases} \|S_0 u\|_E + \left[\sum_{j=1}^\infty \left(2^{js} \|\Delta_j u\|_E \right)^q \right]^{\frac{1}{q}}, & \text{if } 1 \leq q < \infty, \\ \|S_0 u\|_E + \sup_{j \in \mathbb{N}} \left(2^{js} \|\Delta_j u\|_E \right), & \text{if } q = \infty, \end{cases} \quad \forall u \in B_E^{s,q}(\mathbb{R}^N).$$

In particular, if $E = L^p(\mathbb{R}^N)$ ($1 \leq p \leq \infty$) then $B_E^{s,q}(\mathbb{R}^N) = B_{pq}^s(\mathbb{R}^N)$ is the usual Besov space.

Let $\gamma \geq 0$ and $\gamma > s$. It is well-known that $\varphi \in B_E^{s,q}(\mathbb{R}^N)$ if and only if for any $t > 0$ we have $e^{t\Delta} \varphi \in E$ and $t^{-\frac{s}{2} + \frac{\gamma}{2}} \|D^\gamma e^{t\Delta} \varphi\|_E \in L^q((0, t_0), \frac{dt}{t})$, and the norms $\|\varphi\|_{B_E^{s,q}}$ and

$$\|\varphi\|'_{B_E^{s,q}} := \|e^{t_0 \Delta} \varphi\|_E + \left[\int_0^{t_0} \left(t^{-\frac{s}{2} + \frac{\gamma}{2}} \|D^\gamma e^{t\Delta} \varphi\|_E \right)^q \frac{dt}{t} \right]^{\frac{1}{q}}$$

(for $1 \leq q < \infty$) or $\|\varphi\|'_{B_E^{s,q}} := \|e^{t_0 \Delta} \varphi\|_E + \sup_{t \in [0, t_0]} t^{-\frac{s}{2} + \frac{\gamma}{2}} \|D^\gamma e^{t\Delta} \varphi\|_E$ (for $q = \infty$) are equivalent (cf. Theorem 5.3 of [18]). We thus introduce the following concept:

Definition 1.2 *Let E be a shift-invariant Banach space of distributions on \mathbb{R}^N . Let $T > 0$. We denote by $B_E^{-1(\ln), \infty}$ the function space of all distribution $\varphi \in \mathcal{S}'(\mathbb{R}^N)$ such that $e^{t\Delta} \varphi \in E$ for all $t \in (0, \infty)$, and*

$$\|\varphi\|_{B_{E,T}^{-1(\ln), \infty}} := \sup_{t \in (0, T)} \sqrt{t} \left| \ln \left(\frac{t}{e^2 T} \right) \right| \|e^{t\Delta} \varphi\|_E < \infty.$$

In particular, for $E = L^\infty(\mathbb{R}^N)$, the space $B_E^{-1(\ln), \infty}$ is denoted as $B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)$. The notations $\mathcal{C}_{bu}^{-1(\ln)}(\mathbb{R}^N)$ and $\mathcal{C}_0^{-1(\ln)}(\mathbb{R}^N)$ denote the closures of $C_{bu}(\mathbb{R}^N)$ (=the space of bounded and uniformly continuous functions on \mathbb{R}^N) and $C_0^\infty(\mathbb{R}^N)$ in $B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)$, respectively. \square

The definition of $B_E^{-1(\ln), \infty}$ does not depend on the specific choice of T . Indeed, it is easy to check that for any $0 < T_1 < T_2$ we have

$$\|\varphi\|_{B_{E,T_1}^{-1(\ln), \infty}} \leq \|\varphi\|_{B_{E,T_2}^{-1(\ln), \infty}} \leq \sqrt{\frac{T_2}{T_1}} \|\varphi\|_{B_{E,T_1}^{-1(\ln), \infty}}.$$

We note that $B_E^{-1(\ln),\infty}$ is a Banach space. Moreover, we have the following embedding result:

Lemma 1.3 (i) For any $1 < q \leq \infty$, $B_E^{-1(\ln),\infty}$ is continuously embedded into $B_E^{-1,q}$.
(ii) For any $s > -1$, $B_E^{s,\infty}$ is continuously embedded into $B_E^{-1(\ln),\infty}$.

Proof. For $q = \infty$ the assertion (i) is immediate, because $\left| \ln \left(\frac{t}{e^{2T}} \right) \right| \geq 2$ for $0 < t < T$. For $1 < q < \infty$ we deduce as follows:

$$\begin{aligned} \int_0^T \left(t^{\frac{1}{2}} \|e^{t\Delta} \varphi\|_E \right)^q \frac{dt}{t} &= \int_0^T t^{-1} \left| \ln \left(\frac{t}{e^{2T}} \right) \right|^{-q} \cdot \left(\sqrt{t} \left| \ln \left(\frac{t}{e^{2T}} \right) \right| \|e^{t\Delta} \varphi\|_E \right)^q dt \\ &\leq \int_0^T t^{-1} \left| \ln \left(\frac{t}{e^{2T}} \right) \right|^{-q} dt \cdot \left(\sup_{0 < t < T} \sqrt{t} \left| \ln \left(\frac{t}{e^{2T}} \right) \right| \|e^{t\Delta} \varphi\|_E \right)^q \\ &= \frac{2^{1-q}}{q-1} \left(\sup_{0 < t < T} \sqrt{t} \left| \ln \left(\frac{t}{e^{2T}} \right) \right| \|e^{t\Delta} \varphi\|_E \right)^q \end{aligned}$$

Hence $\|\varphi\|_{B_{E,T}^{-1,q}} \leq \left(\frac{2^{1-q}}{q-1} \right)^{\frac{1}{q}} \|\varphi\|_{B_{E,T}^{-1(\ln),\infty}}$. This proves the assertion (i). The assertion (ii) is immediate. \square

Remark It can be easily shown that when $E = L^p(\mathbb{R}^N)$ ($1 \leq p \leq \infty$), $B_E^{-1(\ln),\infty}$ essentially coincides with the space $B_{\infty}^{-1,1}(\mathbb{R}^N)$ introduced by Yoneda in [24].

The main result of this paper is as follows:

Theorem 1.4 Let $0 < r < 1$, $s_r = -1 + r$ and $q_r = \frac{2}{1-r}$. Given $T > 0$, there exist constants $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that for any $\mathbf{u}_0 = \mathbf{v}_0 + \mathbf{w}_0 + \mathbf{z}_0$ with $\mathbf{v}_0 \in B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)$, $\mathbf{w}_0 \in B_{X_r}^{s_r,q_r}(\mathbb{R}^N)$, $\mathbf{z}_0 \in L^2(\mathbb{R}^N)$, $\nabla \cdot \mathbf{v}_0 = \nabla \cdot \mathbf{w}_0 = \nabla \cdot \mathbf{z}_0 = 0$, $\|\mathbf{v}_0\|_{B_{\infty\infty,T}^{-1(\ln)}} < \varepsilon_1$ and $\|\mathbf{w}_0\|_{B_{X_r}^{s_r,q_r}} < \varepsilon_2$, the problem (1.1) has a weak solution $\mathbf{u} \in C_w([0, T], B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N) + B_{X_r}^{s_r,q_r}(\mathbb{R}^N) + L^2(\mathbb{R}^N))$. In particular, for any $\mathbf{u}_0 = \mathbf{v}_0 + \mathbf{w}_0 + \mathbf{z}_0$ with $\mathbf{v}_0 \in \mathcal{C}_0^{-1(\ln)}(\mathbb{R}^N)$, $\mathbf{w}_0 \in B_{\tilde{X}_r}^{s_r,q_r}(\mathbb{R}^N)$, $\mathbf{z}_0 \in L^2(\mathbb{R}^N)$ and $\nabla \cdot \mathbf{v}_0 = \nabla \cdot \mathbf{w}_0 = \nabla \cdot \mathbf{z}_0 = 0$, the problem (1.1) has a weak solution $\mathbf{u} \in C_w([0, T], \mathcal{C}_0^{-1(\ln)}(\mathbb{R}^N) + B_{\tilde{X}_r}^{s_r,q_r}(\mathbb{R}^N) + L^2(\mathbb{R}^N))$.

The idea of the proof of this result is as follows. We first solve the Navier-Stokes equations for initial data $\mathbf{v}_0 \in B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)$ in a regular space. For this purpose we need to establish well-posedness of the Navier-Stokes equations in the space $B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)$. Next we consider the solution of the Navier-Stokes equations of the form $\mathbf{v} + \mathbf{w}$ such that \mathbf{w} satisfies the initial condition $\mathbf{w}_0 \in B_{X_r}^{s_r,q_r}(\mathbb{R}^N)$. This requires to solve a linear perturbation of the Navier-Stokes equations. Generally speaking, such problems are not always solvable due to the nonlinearity of the perturbed equations. Fortunately, since \mathbf{v} is the solution of the Navier-Stokes equations in a regular space, this perturbation problem is solvable and, importantly, the solution also belongs to a regular space. Finally we consider the solution of the Navier-Stokes equations of the form $\mathbf{v} + \mathbf{w} + \mathbf{z}$ such that \mathbf{z} satisfies the initial condition $\mathbf{z}_0 \in L^2(\mathbb{R}^N)$. For this purpose we need to solve another linear perturbation of the Navier-Stokes equations. Thanks to the fact that both

\mathbf{v} and \mathbf{w} are in regular spaces, this perturbation problem is also solvable. But due to the weak regularity of the initial data \mathbf{z}_0 , this time we can only get a weak solution.

The organization of the rest part is as follows. In Section 2 we solve the Navier-Stokes initial value problem for initial data $\mathbf{v}_0 \in B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)$. In Section 3 we solve a linearly perturbed Navier-Stokes initial value problem for initial data $\mathbf{w}_0 \in B_{X_r}^{s_r, q_r}(\mathbb{R}^N)$. In the last section we solve another linearly perturbed Navier-Stokes initial value problem for initial data $\mathbf{z}_0 \in L^2(\mathbb{R}^N)$ and give the proof of Theorem 1.2.

2 Well-posedness in $B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)$ and related spaces

In this section we prove the following two theorems:

Theorem 2.1 *The problem (1.3) is semi-globally weakly well-posed in $B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)$ for small initial data, namely, for any $T > 0$ there exists corresponding constant $\varepsilon > 0$ such that for any $\mathbf{u}_0 \in B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)$ satisfying the conditions $\nabla \cdot \mathbf{u}_0 = 0$ and $\|\mathbf{u}_0\|_{B_{\infty\infty}^{-1(\ln)}} < \varepsilon$, the problem (1.1) has a unique mild solution in the class*

$$\begin{aligned} \mathbf{u} \in C_w([0, T], B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)) \cap L_{\text{loc}}^\infty((0, T), L^\infty(\mathbb{R}^N)), \quad \nabla \cdot \mathbf{u} = 0, \\ \sup_{t \in (0, T)} \sqrt{t} \left| \ln \left(\frac{t}{e^2 T} \right) \right| \|\mathbf{u}(t)\|_\infty < \infty, \end{aligned}$$

and the solution map $\mathbf{u}_0 \mapsto \mathbf{u}$ from $\overline{B}(0, \varepsilon) \subseteq B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)$ to the Banach space of the above class of functions on $\mathbb{R}^N \times [0, T]$ is Lipschitz continuous. Here $C_w([0, T], B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N))$ denotes the Banach space of mappings from $[0, T]$ to $B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)$ which are continuous with respect to the $*$ -weak topology of $B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)$.

Theorem 2.2 (i) *The problem (1.3) is locally well-posed in $C_{bu}^{-1(\ln)}(\mathbb{R}^N)$, namely, for any $\mathbf{u}_0 \in C_{bu}^{-1(\ln)}(\mathbb{R}^N)$ satisfying the condition $\nabla \cdot \mathbf{u}_0 = 0$, there exists corresponding $T > 0$ such that the problem (1.1) has a unique mild solution in the class*

$$\begin{aligned} \mathbf{u} \in C([0, T], C_{bu}^{-1(\ln)}(\mathbb{R}^N)) \cap L_{\text{loc}}^\infty((0, T), L^\infty(\mathbb{R}^N)), \quad \nabla \cdot \mathbf{u} = 0, \\ \sup_{t \in (0, T)} \sqrt{t} \left| \ln \left(\frac{t}{e^2 T} \right) \right| \|\mathbf{u}(t)\|_\infty < \infty, \quad \lim_{t \rightarrow 0^+} \sqrt{t} \left| \ln \left(\frac{t}{e^2 T} \right) \right| \|\mathbf{u}(t)\|_\infty = 0, \end{aligned}$$

and the solution map $\mathbf{u}_0 \mapsto \mathbf{u}$ from $\overline{B}(0, \varepsilon) \subseteq C_{bu}^{-1(\ln)}(\mathbb{R}^N)$ to the Banach space of the above class of functions on $\mathbb{R}^N \times [0, T]$ is Lipschitz continuous.

(ii) *The problem (1.3) is semi-globally well-posed in $C_{bu}^{-1(\ln)}(\mathbb{R}^N)$ for small initial data, namely, for any $T > 0$ there exists corresponding constant $\varepsilon > 0$ such that for any $\mathbf{u}_0 \in C_{bu}^{-1(\ln)}(\mathbb{R}^N)$ satisfying the conditions $\nabla \cdot \mathbf{u}_0 = 0$ and $\|\mathbf{u}_0\|_{B_{\infty\infty}^{-1(\ln)}} < \varepsilon$, the problem (1.1) has a*

unique mild solution in the above class, and the solution map $\mathbf{u}_0 \mapsto \mathbf{u}$ from $\overline{B}(0, \varepsilon) \subseteq C_{bu}^{-1(\ln)}(\mathbb{R}^N)$ to the Banach space of the above class of functions on $\mathbb{R}^N \times [0, T]$ is Lipschitz continuous.

To prove Theorem 2.1 we shall work in the path space

$$\mathcal{X}_T = \{\mathbf{u} \in L_{\text{loc}}^\infty((0, T), L^\infty(\mathbb{R}^N)) : \nabla \cdot \mathbf{u} = 0, \quad \|\mathbf{u}\|_{\mathcal{X}_T} = \sup_{0 < t < T} \sqrt{t} \left| \ln \left(\frac{t}{e^2 T} \right) \right| \|\mathbf{u}(t)\|_\infty < \infty\}.$$

$(\mathcal{X}_T, \|\cdot\|_{\mathcal{X}_T})$ is a Banach space. The solution will lie in the space

$$\mathcal{Y}_T = C_w([0, T], B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)) \cap \mathcal{X}_T.$$

To prove Theorem 2.2 we shall work in the path space

$$\mathcal{X}_T^0 = \{\mathbf{u} \in \mathcal{X}_T : \lim_{t \rightarrow 0^+} \sqrt{t} \left| \ln \left(\frac{t}{e^2 T} \right) \right| \|\mathbf{u}(t)\|_\infty = 0\};$$

the solution will lie in the space

$$\mathcal{Y}_T^0 = C([0, T], C_{bu}^{-1(\ln)}(\mathbb{R}^N)) \cap \mathcal{X}_T^0.$$

We first prove some preliminary lemmas. In what follows, the expression $A \lesssim_T B$ means $A \leq C_T B$ for some constant C_T depending on T ; if $A \leq C B$ for some constant C independent of T then we write $A \lesssim B$.

Lemma 2.3 *If $\varphi \in B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)$ then $e^{t\Delta}\varphi \in \mathcal{Y}_T$ for any finite $T > 0$, and*

$$\|e^{t\Delta}\varphi\|_{\mathcal{X}_T} + \sup_{t \in (0, T)} \|e^{t\Delta}\varphi\|_{B_{\infty\infty}^{-1(\ln)}} \lesssim_T \|\varphi\|_{B_{\infty\infty}^{-1(\ln)}}.$$

If furthermore $\varphi \in C_{bu}^{-1(\ln)}(\mathbb{R}^N)$ then in addition to the above estimate we also have $e^{t\Delta}\varphi \in \mathcal{Y}_T^0$, and

$$\lim_{T \rightarrow 0^+} \|e^{t\Delta}\varphi\|_{\mathcal{X}_T} = 0.$$

Proof: It is easy to see that $B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)$ is a shift-invariant Banach space of distributions. Hence by Propositions 4.1 and 4.4 of [18] we see that $\varphi \in B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)$ implies that $e^{t\Delta}\varphi \in C_*([0, T], B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N))$, i.e., the map $t \mapsto e^{t\Delta}\varphi$ from $[0, T]$ to $B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)$ is continuous for $0 < t \leq T$ with respect to the norm topology of $B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)$ and continuous at $t = 0$ with respect to the $*$ -weak topology of $B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)$, and

$$\sup_{t \in [0, T]} \|e^{t\Delta}\varphi\|_{B_{\infty\infty}^{-1(\ln)}} \leq \|\varphi\|_{B_{\infty\infty}^{-1(\ln)}}.$$

Moreover, from the definition of the space $B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)$ we see that $\varphi \in B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)$ implies that $e^{t\Delta}\varphi \in \mathcal{X}_T$, and

$$\|e^{t\Delta}\varphi\|_{\mathcal{X}_T} \lesssim_T \|\varphi\|_{B_{\infty\infty}^{-1(\ln)}}.$$

Hence the first part of the lemma follows. Next we assume that $\varphi \in \mathcal{C}_{bu}^{-1(\ln)}(\mathbb{R}^N)$. We choose a sequence of functions $\psi_n \in C_{bu}(\mathbb{R}^N)$ ($n = 1, 2, \dots$) such that $\lim_{n \rightarrow \infty} \|\varphi - \psi_n\|_{B_{\infty\infty}^{-1(\ln)}} = 0$. We write

$$\begin{aligned} \|e^{t\Delta}\varphi - \varphi\|_{B_{\infty\infty}^{-1(\ln)}} &\leq \|e^{t\Delta}\varphi - e^{t\Delta}\psi_n\|_{B_{\infty\infty}^{-1(\ln)}} + \|e^{t\Delta}\psi_n - \psi_n\|_{B_{\infty\infty}^{-1(\ln)}} + \|\psi_n - \varphi\|_{B_{\infty\infty}^{-1(\ln)}} \\ &\leq 2\|\varphi - \psi_n\|_{B_{\infty\infty}^{-1(\ln)}} + C\|e^{t\Delta}\psi_n - \psi_n\|_{\infty}. \end{aligned}$$

Since $e^{t\Delta}$ is a C_0 -semigroup in $C_{bu}(\mathbb{R}^N)$, we have $\lim_{t \rightarrow 0^+} \|e^{t\Delta}\psi_n - \psi_n\|_{\infty} = 0$ (for each n). Hence, from the above estimate we immediately see that $\lim_{t \rightarrow 0^+} \|e^{t\Delta}\varphi - \varphi\|_{B_{\infty\infty}^{-1(\ln)}} = 0$, so that $e^{t\Delta}\varphi$ is also continuous at $t = 0$. Therefore, $e^{t\Delta}\varphi \in C([0, T], \mathcal{C}_{bu}^{-1(\ln)}(\mathbb{R}^N))$. Moreover, for any $\psi \in C_{bu}(\mathbb{R}^N)$ we can easily check that $e^{t\Delta}\psi \in \mathcal{X}_T^0$ and $\lim_{T \rightarrow 0^+} \|e^{t\Delta}\psi\|_{\mathcal{X}_T} = 0$. From this assertion and the fact that

$$\|e^{t\Delta}\varphi - e^{t\Delta}\psi_n\|_{\mathcal{X}_T} \lesssim \|\varphi - \psi_n\|_{B_{\infty\infty}^{-1(\ln)}}$$

(uniformly for $0 < T \leq 1$) we see that also $e^{t\Delta}\varphi \in \mathcal{X}_T^0$ and $\lim_{T \rightarrow 0^+} \|e^{t\Delta}\varphi\|_{\mathcal{X}_T} = 0$. This completes the proof of the lemma. \square

In what follows we consider the bilinear map:

$$B(\mathbf{u}, \mathbf{v})(t) = \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot [\mathbf{u}(\tau) \otimes \mathbf{v}(\tau)] d\tau.$$

The following lemma will play a fundamental role:

Lemma 2.4 *Let E be a shift-invariant Banach space of distributions. Let \mathbf{u} be a $N \times N$ matrix function on \mathbb{R}^N with all entries belonging to E . Then*

$$\|e^{t\Delta} \mathbb{P} \nabla \cdot \mathbf{u}\|_E \lesssim t^{-\frac{1}{2}} \|\mathbf{u}\|_E, \quad \text{for all } t > 0.$$

Proof. In the case $E = L^p(\mathbb{R}^N)$ ($1 \leq p \leq \infty$), this inequality was proved by Giga, Inui and Matsui in [11]. The general case follows from a similar argument. Indeed, by Proposition 11.1 of [18], the kernel of the convolution operator $e^{t\Delta} \mathbb{P} \nabla$ belongs to $L^1(\mathbb{R}^N)$ and its L^1 norm is bounded by $C\sqrt{t}$. Hence the above inequality follows from the shift-invariance of E . \square

We now establish estimates on the bilinear map $B(\mathbf{u}, \mathbf{v})$.

Lemma 2.5 *Let $T > 0$ be given and assume that $\mathbf{u}, \mathbf{v} \in \mathcal{X}_T$. Then $B(\mathbf{u}, \mathbf{v}) \in \mathcal{X}_T \cap C_w([0, T], B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N))$, and*

$$\|B(\mathbf{u}, \mathbf{v})\|_{\mathcal{X}_T} \lesssim \|\mathbf{u}\|_{\mathcal{X}_T} \|\mathbf{v}\|_{\mathcal{X}_T}, \tag{2.1}$$

$$\sup_{t \in (0, T)} \|B(\mathbf{u}, \mathbf{v})(t)\|_{B_{\infty\infty}^{-1(\ln)}} \lesssim \|\mathbf{u}\|_{\mathcal{X}_T} \|\mathbf{v}\|_{\mathcal{X}_T}. \tag{2.2}$$

$$\lim_{t \rightarrow 0^+} B(\mathbf{u}, \mathbf{v})(t) = 0 \quad (*\text{-weakly in } B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)). \tag{2.3}$$

If furthermore either $\mathbf{u} \in \mathcal{B}_T^0$ or $\mathbf{v} \in \mathcal{B}_T^0$ then also $B(\mathbf{u}, \mathbf{v}) \in \mathcal{B}_T^0$, and

$$\lim_{t \rightarrow 0^+} \|B(\mathbf{u}, \mathbf{v})(t)\|_{B_{\infty\infty}^{-1}(\ln)} = 0. \quad (2.4)$$

Proof: By Lemma 2.4 (choose $E = L^\infty(\mathbb{R}^N)$) we have

$$\begin{aligned} \|B(\mathbf{u}, \mathbf{v})(t)\|_\infty &\lesssim \int_0^t (t-\tau)^{-\frac{1}{2}} \|\mathbf{u}(\tau) \otimes \mathbf{v}(\tau)\|_\infty d\tau \\ &\lesssim \sup_{0 < \tau < t} \sqrt{\tau} \left| \ln \left(\frac{\tau}{e^2 T} \right) \right| \|\mathbf{u}(\tau)\|_\infty \cdot \sup_{0 < \tau < t} \sqrt{\tau} \left| \ln \left(\frac{\tau}{e^2 T} \right) \right| \|\mathbf{v}(\tau)\|_\infty \\ &\quad \times \int_0^t (t-\tau)^{-\frac{1}{2}} \tau^{-1} \left| \ln \left(\frac{\tau}{e^2 T} \right) \right|^{-2} d\tau. \end{aligned} \quad (2.5)$$

It is easy to show that

$$\int_0^t (t-\tau)^{-\frac{1}{2}} \tau^{-1} \left| \ln \left(\frac{\tau}{e^2 T} \right) \right|^{-2} d\tau \lesssim t^{-\frac{1}{2}} \left| \ln \left(\frac{t}{e^2 T} \right) \right|^{-1}. \quad (2.6)$$

Indeed, we have

$$\begin{aligned} &\int_0^t (t-\tau)^{-\frac{1}{2}} \tau^{-1} \left| \ln \left(\frac{\tau}{e^2 T} \right) \right|^{-2} d\tau \\ &= \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{1}{2}} \tau^{-1} \left| \ln \left(\frac{\tau}{e^2 T} \right) \right|^{-2} d\tau + \int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1}{2}} \tau^{-1} \left| \ln \left(\frac{\tau}{e^2 T} \right) \right|^{-2} d\tau \\ &\lesssim \sqrt{\frac{2}{t}} \int_0^{\frac{t}{2}} \tau^{-1} \left| \ln \left(\frac{\tau}{e^2 T} \right) \right|^{-2} d\tau + \left(\frac{t}{2} \right)^{-1} \left| \ln \left(\frac{t}{2e^2 T} \right) \right|^{-2} \int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1}{2}} d\tau \\ &= \left(\frac{t}{2} \right)^{-\frac{1}{2}} \left| \ln \left(\frac{t}{2e^2 T} \right) \right|^{-1} + \sqrt{2t} \left(\frac{t}{2} \right)^{-1} \left| \ln \left(\frac{t}{2e^2 T} \right) \right|^{-2} \\ &\lesssim t^{-\frac{1}{2}} \left| \ln \left(\frac{t}{e^2 T} \right) \right|^{-1}. \end{aligned}$$

Hence $B(\mathbf{u}, \mathbf{v})(t) \in \mathcal{X}_T$ and (2.1) holds. Next, we note that also by Lemma 2.4, for any $s \in [0, T]$ and $t \in [0, T]$ we have

$$\begin{aligned} \|e^{s\Delta} B(\mathbf{u}, \mathbf{v})(t)\|_\infty &= \left\| \int_0^t e^{(t+s-\tau)\Delta} \mathbb{P} \nabla \cdot [\mathbf{u}(\tau) \otimes \mathbf{v}(\tau)] d\tau \right\|_\infty \\ &\lesssim \int_0^t (t+s-\tau)^{-\frac{1}{2}} \|\mathbf{u}(\tau) \otimes \mathbf{v}(\tau)\|_\infty d\tau \\ &\lesssim \sup_{0 < \tau < t} \sqrt{\tau} \left| \ln \left(\frac{\tau}{e^2 T} \right) \right| \|\mathbf{u}(\tau)\|_\infty \cdot \sup_{0 < \tau < t} \sqrt{\tau} \left| \ln \left(\frac{\tau}{e^2 T} \right) \right| \|\mathbf{v}(\tau)\|_\infty \\ &\quad \times \int_0^t (t+s-\tau)^{-\frac{1}{2}} \tau^{-1} \left| \ln \left(\frac{\tau}{e^2 T} \right) \right|^{-2} d\tau. \end{aligned}$$

By (2.6) we have

$$\begin{aligned}
\int_0^t (t+s-\tau)^{-\frac{1}{2}} \tau^{-1} \left| \ln \left(\frac{\tau}{e^2 T} \right) \right|^{-2} d\tau &\lesssim \int_0^{t+s} (t+s-\tau)^{-\frac{1}{2}} \tau^{-1} \left| \ln \left(\frac{\tau}{e^2 T} \right) \right|^{-2} d\tau \\
&\lesssim (t+s)^{-\frac{1}{2}} \left| \ln \left(\frac{t+s}{e^2 T} \right) \right|^{-1} \\
&\lesssim s^{-\frac{1}{2}} \left| \ln \left(\frac{s}{e^2 T} \right) \right|^{-1}.
\end{aligned}$$

Hence for any $t \in [0, T]$ we have

$$\begin{aligned}
\|B(\mathbf{u}, \mathbf{v})(t)\|_{B_{\infty\infty}^{-1}(\ln)} &= \sup_{0 < s < T} \sqrt{s} \left| \ln \left(\frac{s}{e^2 T} \right) \right| \|e^{s\Delta} B(\mathbf{u}, \mathbf{v})(t)\|_{\infty} \\
&\lesssim \sup_{0 < \tau < t} \sqrt{\tau} \left| \ln \left(\frac{\tau}{e^2 T} \right) \right| \|\mathbf{u}(\tau)\|_{\infty} \cdot \sup_{0 < \tau < t} \sqrt{\tau} \left| \ln \left(\frac{\tau}{e^2 T} \right) \right| \|\mathbf{v}(\tau)\|_{\infty} \quad (2.7) \\
&\lesssim \|\mathbf{u}\|_{\mathcal{X}_T} \|\mathbf{v}\|_{\mathcal{X}_T}.
\end{aligned}$$

This proves (2.2).

The proof of (2.3) is easy. Indeed, for any $\mathbf{w} \in \mathcal{S}(\mathbb{R}^N)$ we have

$$\begin{aligned}
|\langle B(\mathbf{u}, \mathbf{v})(t), \mathbf{w} \rangle| &= \left| \int_0^t \langle e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot [\mathbf{u}(\tau) \otimes \mathbf{v}(\tau)], \mathbf{w} \rangle d\tau \right| \\
&\leq \int_0^t \|\mathbf{u}(\tau)\|_{\infty} \|\mathbf{v}(\tau)\|_{\infty} \|\mathbb{P} \nabla \mathbf{w}\|_1 d\tau \\
&\leq \|\mathbf{u}\|_{\mathcal{X}_T} \|\mathbf{v}\|_{\mathcal{X}_T} \|\mathbb{P} \nabla \mathbf{w}\|_1 \int_0^t \tau^{-1} \left| \ln \left(\frac{\tau}{e^2 T} \right) \right|^{-2} d\tau \\
&= \left| \ln \left(\frac{t}{e^2 T} \right) \right|^{-1} \|\mathbf{u}\|_{\mathcal{X}_T} \|\mathbf{v}\|_{\mathcal{X}_T} \|\mathbb{P} \nabla \mathbf{w}\|_1 \rightarrow 0 \quad (\text{as } t \rightarrow 0^+).
\end{aligned}$$

Since $\mathcal{S}(\mathbb{R}^N)$ is dense in the predual of $B_{\infty\infty}^{-1}(\ln)(\mathbb{R}^N)$, using this result and (2.2) we obtain (2.3).

We now proceed to prove that $B(\mathbf{u}, \mathbf{v}) \in C_w([0, T], B_{\infty\infty}^{-1}(\ln)(\mathbb{R}^N))$. By (2.3), $B(\mathbf{u}, \mathbf{v})(t)$ is $*$ -weakly continuous at $t = 0$. We now consider an arbitrary point $0 < t_0 \leq T$. If $t_0 < t < T$ then we write

$$\begin{aligned}
&B(\mathbf{u}, \mathbf{v})(t) - B(\mathbf{u}, \mathbf{v})(t_0) \\
&= \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot [\mathbf{u}(\tau) \otimes \mathbf{v}(\tau)] d\tau - \int_0^{t_0} e^{(t_0-\tau)\Delta} \mathbb{P} \nabla \cdot [\mathbf{u}(\tau) \otimes \mathbf{v}(\tau)] d\tau \\
&= \int_{t_0}^t e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot [\mathbf{u}(\tau) \otimes \mathbf{v}(\tau)] d\tau + [e^{(t-t_0)\Delta} - I] \int_0^{t_0} e^{(t_0-\tau)\Delta} \mathbb{P} \nabla \cdot [\mathbf{u}(\tau) \otimes \mathbf{v}(\tau)] d\tau \\
&=: A(t) + B(t), \tag{2.8}
\end{aligned}$$

and if $0 < t_0 - \delta < t < t_0$ then we write

$$\begin{aligned}
& B(\mathbf{u}, \mathbf{v})(t_0) - B(\mathbf{u}, \mathbf{v})(t) \\
&= \int_0^{t_0} e^{(t_0-\tau)\Delta} \mathbb{P} \nabla \cdot [\mathbf{u}(\tau) \otimes \mathbf{v}(\tau)] d\tau - \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot [\mathbf{u}(\tau) \otimes \mathbf{v}(\tau)] d\tau \\
&= \int_t^{t_0} e^{(t_0-\tau)\Delta} \mathbb{P} \nabla \cdot [\mathbf{u}(\tau) \otimes \mathbf{v}(\tau)] d\tau + [e^{(t_0-t)\Delta} - I] \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot [\mathbf{u}(\tau) \otimes \mathbf{v}(\tau)] d\tau \\
&= \int_t^{t_0} e^{(t_0-\tau)\Delta} \mathbb{P} \nabla \cdot [\mathbf{u}(\tau) \otimes \mathbf{v}(\tau)] d\tau + [e^{(t_0-t)\Delta} - I] \int_{t_0-\delta}^t e^{(t-\tau)\Delta} \mathbb{P} \nabla \cdot [\mathbf{u}(\tau) \otimes \mathbf{v}(\tau)] d\tau \\
&\quad + e^{(t-t_0+\delta)\Delta} [e^{(t_0-t)\Delta} - I] \int_0^{t_0-\delta} e^{(t_0-\delta-\tau)\Delta} \mathbb{P} \nabla \cdot [\mathbf{u}(\tau) \otimes \mathbf{v}(\tau)] d\tau \\
&=: A_1(t) + B_1(t) + B_2(t).
\end{aligned} \tag{2.9}$$

For $A(t)$ we have (see the proof of (2.7))

$$\begin{aligned}
\|A(t)\|_{B_{\infty\infty}^{-1(\ln)}} &= \sup_{0 < s < T} \sqrt{s} \left| \ln \left(\frac{s}{e^{2T}} \right) \right| \left\| \int_{t_0}^t e^{(t+s-\tau)\Delta} \mathbb{P} \nabla \cdot [\mathbf{u}(\tau) \otimes \mathbf{v}(\tau)] d\tau \right\|_{\infty} \\
&\lesssim \|\mathbf{u}\|_{\mathcal{X}_T} \|\mathbf{v}\|_{\mathcal{X}_T} \sup_{0 < s < T} \sqrt{s} \left| \ln \left(\frac{s}{e^{2T}} \right) \right| \int_{t_0}^t (t+s-\tau)^{-\frac{1}{2}} \tau^{-1} \left| \ln \left(\frac{\tau}{e^{2T}} \right) \right|^{-2} d\tau \\
&\lesssim_T \|\mathbf{u}\|_{\mathcal{X}_T} \|\mathbf{v}\|_{\mathcal{X}_T} \int_{t_0}^t (t-\tau)^{-\frac{1}{2}} \tau^{-1} \left| \ln \left(\frac{\tau}{e^{2T}} \right) \right|^{-2} d\tau,
\end{aligned}$$

so that $\lim_{t \rightarrow t_0^+} \|A(t)\|_{B_{\infty\infty}^{-1(\ln)}} = 0$. Moreover, since $B(t) = [e^{(t-t_0)\Delta} - I]B(\mathbf{u}, \mathbf{v})(t_0)$ and $B(\mathbf{u}, \mathbf{v})(t_0) \in B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)$, by Lemma 2.3 we have $\lim_{t \rightarrow t_0^+} B(t) = 0$ ($*$ -weakly in $B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)$). Hence

$$\lim_{t \rightarrow t_0^+} B(\mathbf{u}, \mathbf{v})(t) = B(\mathbf{u}, \mathbf{v})(t_0) \quad (*\text{-weakly in } B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)).$$

Next, similarly as for $A(t)$ we have $\lim_{t \rightarrow t_0^-} \|A_1(t)\|_{B_{\infty\infty}^{-1(\ln)}} = 0$. Moreover, similarly as for the treatment of $A(t)$ we have that by choosing δ sufficiently small, $\|B_1(t)\|_{B_{\infty\infty}^{-1(\ln)}}$ can be as small as we expect, and when δ is chosen and fixed, $B_2(t)$ can be treated similarly as for $B(t)$ to get that for any \mathbf{w} in the predual of $B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)$, $\lim_{t \rightarrow t_0^-} \langle B_2(t), \mathbf{w} \rangle = 0$. Hence

$$\lim_{t \rightarrow t_0^-} B(\mathbf{u}, \mathbf{v})(t) = B(\mathbf{u}, \mathbf{v})(t_0) \quad (*\text{-weakly in } B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)).$$

This proves $B(\mathbf{u}, \mathbf{v}) \in C_w([0, T], B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N))$.

We now furthermore assume that either $\mathbf{u} \in \mathcal{X}_T^0$ or $\mathbf{v} \in \mathcal{X}_T^0$. It is not difficult to see that in this case we have $B(\mathbf{u}, \mathbf{v})(t) \in \mathcal{C}_{bu}^{-1(\ln)}(\mathbb{R}^N)$ for each fixed $t \in [0, T]$, and from (2.7) we see that (2.4) holds. Moreover, since $e^{t\Delta}$ is a C_0 -semigroup in $C_{bu}(\mathbb{R}^N)$ and $B(\mathbf{u}, \mathbf{v})(t_0) \in \mathcal{C}_{bu}^{-1(\ln)}(\mathbb{R}^N)$

for any $0 < t_0 \leq T$, from the definition of $B(t)$ (see (2.8)) we see that $\lim_{t \rightarrow t_0^+} \|B(t)\|_{B_{\infty\infty}^{-1}(\ln)} = 0$, so that

$$\lim_{t \rightarrow t_0^+} \|B(\mathbf{u}, \mathbf{v})(t) - B(\mathbf{u}, \mathbf{v})(t_0)\|_{B_{\infty\infty}^{-1}(\ln)} = 0$$

for any $0 < t_0 < T$. Similarly we have

$$\lim_{t \rightarrow t_0^-} \|B(\mathbf{u}, \mathbf{v})(t) - B(\mathbf{u}, \mathbf{v})(t_0)\|_{B_{\infty\infty}^{-1}(\ln)} = 0.$$

for any $0 < t_0 \leq T$. Hence $B(\mathbf{u}, \mathbf{v}) \in C([0, T], C_{bu}^{-1}(\ln)(\mathbb{R}^N))$. Finally, from (2.5) and (2.6) we see that $B(\mathbf{u}, \mathbf{v}) \in \mathcal{X}_T^0$. It follows that $B(\mathbf{u}, \mathbf{v}) \in \mathcal{Y}_T^0$. This completes the proof of Lemma 2.6. \square

Having proved Lemmas 2.3 and 2.5, Theorems 2.1 and 2.2 follow from a standard fixed point argument. Indeed, we rewrite the problem (1.3) into the following equivalent integral equation:

$$\mathbf{u}(t) = e^{t\Delta}\mathbf{u}_0 - B(\mathbf{u}, \mathbf{u})(t).$$

We define a map \mathcal{F} such that for given \mathbf{u} , $\mathcal{F}(\mathbf{u})$ equals to the right-hand side of the above equation. By Lemmas 2.3 and 2.5, if $\mathbf{u}_0 \in B_{\infty\infty}^{-1}(\ln)(\mathbb{R}^N)$ then \mathcal{F} maps \mathcal{X}_T into $\mathcal{Y}_T \subseteq \mathcal{X}_T$. Furthermore, we have the following estimates:

$$\begin{aligned} \|\mathcal{F}(\mathbf{u})\|_{\mathcal{X}_T} &\leq \|e^{t\Delta}\mathbf{u}_0\|_{\mathcal{X}_T} + C\|\mathbf{u}\|_{\mathcal{X}_T}^2, \\ \|\mathcal{F}(\mathbf{u}) - \mathcal{F}(\mathbf{v})\|_{\mathcal{X}_T} &\leq C(\|\mathbf{u}\|_{\mathcal{X}_T} + \|\mathbf{v}\|_{\mathcal{X}_T})\|\mathbf{u} - \mathbf{v}\|_{\mathcal{X}_T}. \end{aligned}$$

To prove the assertion (i) of Theorem 2.1, for given $T > 0$ we let $\mathbf{u}_0 \in B_{\infty 2}^{-1}(\mathbb{R}^N)$ (with $\operatorname{div} \mathbf{u}_0 = 0$) be so small that $\|e^{t\Delta}\mathbf{u}_0\|_{\mathcal{X}_T} \leq \varepsilon$, where ε is an arbitrarily chosen positive number such that $4C\varepsilon < 1$. Then from the first inequality we easily see that \mathcal{F} maps the closed ball $\overline{B}(0, 2\varepsilon)$ in \mathcal{X}_T into itself, and the second inequality ensures that \mathcal{F} is a contraction mapping when restricted to this ball. Hence, by the fixed point theorem of Banach, \mathcal{F} has a unique fixed point in this ball. Since $e^{t\Delta}\mathbf{u}_0 \in \mathcal{Y}_T$ and $\mathcal{F}(\mathcal{X}_T) \subseteq \mathcal{Y}_T$, we get a mild solution of the problem (1.1) which lies in the space \mathcal{Y}_T . This proves Theorem 2.1. The proof of Theorem 2.2 is similar, and we omit it here.

3 Linearly perturbed Navier-Stokes equations

We now consider the following initial value problem:

$$\begin{cases} \partial_t \mathbf{u} - \Delta \mathbf{u} + \mathbb{P} \nabla \cdot (2\mathbf{a} \otimes_s \mathbf{u} + \mathbf{u} \otimes \mathbf{u}) = 0, & x \in \mathbb{R}^N, \ t > 0, \\ \operatorname{div} \mathbf{u} = 0, & x \in \mathbb{R}^N, \ t > 0, \\ \mathbf{u}(x, 0) = \mathbf{u}_0(x), & x \in \mathbb{R}^N, \end{cases} \quad (3.1)$$

where $\mathbf{a} = \mathbf{a}(x, t)$ is a given vector function in $\mathbb{R}^N \times \mathbb{R}_+$, and $\mathbf{a} \otimes_s \mathbf{u} = \frac{1}{2}(\mathbf{a} \otimes \mathbf{u} + \mathbf{u} \otimes \mathbf{a})$.

In this section we study well-posedness of the above problem. We shall not consider the most general situation, but for the purpose of our later application we only consider the case where the initial data belongs to $B_{\tilde{X}_r}^{s_r, q_r}(\mathbb{R}^N)$ for some $0 < r < 1$, where $s_r = -1 + r$ and $q_r = \frac{2}{1-r}$.

Theorem 3.1 *Let $0 < T < \infty$, \mathbf{a} and \mathbf{u}_0 be given such that $\sqrt{t}\mathbf{a}(t) \in L^\infty(\mathbb{R}^N \times [0, T])$ and $\mathbf{u}_0 \in B_{\tilde{X}_r}^{s_r, q_r}(\mathbb{R}^N)$, where $0 < r < 1$, $s_r = -1 + r$ and $q_r = \frac{2}{1-r}$. Assume that $\lim_{t \rightarrow 0^+} \sqrt{t}\|\mathbf{a}(t)\|_\infty = 0$. Then we have the following assertions:*

(i) *Given \mathbf{a} and \mathbf{u}_0 as above, there exists corresponding $T_1 = T_1(\mathbf{a}, \mathbf{u}_0) \in (0, T]$ such that the problem (3.1) has a unique mild solution in the class*

$$\begin{aligned} \mathbf{u} &\in C([0, T_1], B_{\tilde{X}_r}^{s_r, q_r}(\mathbb{R}^N)) \cap L^{q_r}([0, T_1], X_r(\mathbb{R}^N)), \quad \nabla \cdot \mathbf{u} = 0, \\ t^{\frac{1-r}{2}} \mathbf{u}(t) &\in L^\infty([0, T_1], X_r(\mathbb{R}^N)), \quad \sqrt{t}\mathbf{u}(t) \in L^\infty([0, T_1], L^\infty(\mathbb{R}^N)), \\ \lim_{t \rightarrow 0^+} t^{\frac{1-r}{2}} \|\mathbf{u}(t)\|_{X_r} &= 0, \quad \lim_{t \rightarrow 0^+} \sqrt{t}\|\mathbf{u}(t)\|_\infty = 0. \end{aligned}$$

Moreover, the solution map $\mathbf{u}_0 \mapsto \mathbf{u}$ from a small neighborhood of any given point \mathbf{v}_0 of $B_{\tilde{X}_r}^{s_r, q_r}(\mathbb{R}^N)$ to the Banach space of functions of the above class (with $T_1 = T_1(\mathbf{a}, \mathbf{v}_0)$) is Lipschitz continuous. Besides, if $\mathbf{u}_0 \in B_{\tilde{X}_r}^{s_r, q_r}(\mathbb{R}^N)$ then $\mathbf{u} \in C([0, T_1], B_{\tilde{X}_r}^{s_r, q_r}(\mathbb{R}^N))$.

(ii) *Given $T > 0$, there exist corresponding constants $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ depending only on T and N , such that if $\sup_{t \in [0, T]} \sqrt{t}\|\mathbf{a}(t)\|_\infty < \varepsilon_1$ and $\|\mathbf{u}_0\|_{B_{\tilde{X}_r}^{s_r, q_r}} < \varepsilon_2$, the problem (3.1) has a unique mild solution in the class*

$$\begin{aligned} \mathbf{u} &\in C([0, T], B_{\tilde{X}_r}^{s_r, q_r}(\mathbb{R}^N)) \cap L^{q_r}([0, T], X_r(\mathbb{R}^N)), \quad \nabla \cdot \mathbf{u} = 0, \\ t^{\frac{1-r}{2}} \mathbf{u}(t) &\in L^\infty([0, T], X_r(\mathbb{R}^N)), \quad \sqrt{t}\mathbf{u}(t) \in L^\infty([0, T], L^\infty(\mathbb{R}^N)), \\ \lim_{t \rightarrow 0^+} t^{\frac{1-r}{2}} \|\mathbf{u}(t)\|_{X_r} &= 0, \quad \lim_{t \rightarrow 0^+} \sqrt{t}\|\mathbf{u}(t)\|_\infty = 0. \end{aligned}$$

Moreover, the solution map $\mathbf{u}_0 \mapsto \mathbf{u}$ from the ball $\overline{B}(0, \varepsilon)$ of $B_{\tilde{X}_r}^{s_r, q_r}(\mathbb{R}^N)$ to the Banach space of functions of the above class is Lipschitz continuous. Besides, if $\mathbf{u}_0 \in B_{\tilde{X}_r}^{s_r, q_r}(\mathbb{R}^N)$ then $\mathbf{u} \in C([0, T], B_{\tilde{X}_r}^{s_r, q_r}(\mathbb{R}^N))$.

To prove this result, we rewrite the problem (3.1) into the following equivalent integral equation:

$$\mathbf{u}(t) = e^{t\Delta} \mathbf{u}_0 - B(\mathbf{a}, \mathbf{u}) - B(\mathbf{u}, \mathbf{a}) - B(\mathbf{u}, \mathbf{u}). \quad (3.2)$$

To prove existence of a solution to this equation, we shall work in the path space

$$\begin{aligned} \mathcal{V}_T = \Big\{ \mathbf{u} \in L^{q_r}([0, T], X_r(\mathbb{R}^N)) : \sup_{t \in [0, T]} t^{\frac{1-r}{2}} \|\mathbf{u}(t)\|_{X_r} < \infty, \sup_{t \in [0, T]} \sqrt{t}\|\mathbf{u}(t)\|_\infty < \infty, \\ \lim_{t \rightarrow 0^+} t^{\frac{1-r}{2}} \|\mathbf{u}(t)\|_{X_r} = 0, \lim_{t \rightarrow 0^+} \sqrt{t}\|\mathbf{u}(t)\|_\infty = 0 \Big\}, \end{aligned}$$

which is a Banach space with respect to the norm

$$\|u\|_{\mathcal{V}_T} = \|u\|_{L_T^{q_r} X_r} + \sup_{t \in [0, T]} t^{\frac{1-r}{2}} \|u(t)\|_{X_r} + \sup_{t \in [0, T]} \sqrt{t} \|u(t)\|_{\infty},$$

where $\|u\|_{L_T^{q_r} X_r} = \left(\int_0^T \|u(t)\|_{X_r}^{q_r} dt \right)^{\frac{1}{q_r}}$. We shall prove that the solution lies in the space

$$\mathcal{W}_T = C([0, T], B_{X_r}^{s_r, q_r}(\mathbb{R}^N)) \cap \mathcal{V}_T.$$

As in Section 2, the proof will be based on several preliminary Lemmas.

Lemma 3.2 $\{e^{t\Delta}\}_{t \geq 0}$ is a C_0 -semigroup in $B_{X_r}^{s_r, q_r}(\mathbb{R}^N)$, and for any $\varphi \in B_{X_r}^{s_r, q_r}(\mathbb{R}^N)$ we have $e^{t\Delta}\varphi \in \mathcal{W}_T$, and

$$\|e^{t\Delta}\varphi\|_{\mathcal{W}_T} \lesssim_T \|\varphi\|_{B_{X_r}^{s_r, q_r}}.$$

Moreover, if $\varphi \in B_{\tilde{X}_r}^{s_r, q_r}(\mathbb{R}^N)$ then $e^{t\Delta}\varphi \in C([0, T], B_{\tilde{X}_r}^{s_r, q_r}(\mathbb{R}^N))$.

Proof. Since $B_{X_r}^{s_r, q_r}(\mathbb{R}^N)$ is a shift-invariant Banach space of distributions, for any $\varphi \in B_{X_r}^{s_r, q_r}(\mathbb{R}^N)$ we have

$$\|e^{t\Delta}\varphi\|_{B_{X_r}^{s_r, q_r}} \lesssim_T \|\varphi\|_{B_{X_r}^{s_r, q_r}},$$

and $e^{t\Delta}\varphi \in C_*([0, T], B_{X_r}^{s_r, q_r}(\mathbb{R}^N))$. Moreover, the heat kernel characterization of $B_{X_r}^{s_r, q_r}(\mathbb{R}^N)$ implies that

$$\|e^{t\Delta}\varphi\|_{L_T^{q_r} X_r} \lesssim_T \|\varphi\|_{B_{X_r}^{s_r, q_r}},$$

the embedding $B_{X_r}^{s_r, q_r}(\mathbb{R}^N) \subseteq B_{X_r}^{s_r, \infty}(\mathbb{R}^N)$ and the heat kernel characterization of $B_{X_r}^{s_r, \infty}(\mathbb{R}^N)$ implies that

$$\sup_{t \in [0, T]} t^{\frac{1-r}{2}} \|e^{t\Delta}\varphi\|_{X_r} \lesssim_T \|\varphi\|_{B_{X_r}^{s_r, q_r}},$$

and the embedding $B_{X_r}^{s_r, q_r}(\mathbb{R}^N) \hookrightarrow B_{\infty\infty}^{-1}(\mathbb{R}^N)$ and the heat kernel characterization of $B_{\infty\infty}^{-1}(\mathbb{R}^N)$ implies that

$$\sup_{t \in [0, T]} \sqrt{t} \|e^{t\Delta}\varphi\|_{\infty} \lesssim \|\varphi\|_{B_{X_r}^{s_r, q_r}}.$$

The embedding $B_{X_r}^{s_r, q_r}(\mathbb{R}^N) \hookrightarrow B_{\infty\infty}^{-1}(\mathbb{R}^N)$ follows from the embedding $X_r(\mathbb{R}^N) \hookrightarrow B_{\infty\infty}^{-r}(\mathbb{R}^N)$, whose proof can be found in [9].

Next we prove that for any $\varphi \in B_{X_r}^{s_r, q_r}(\mathbb{R}^N)$, $\lim_{t \rightarrow 0^+} \|e^{t\Delta}\varphi - \varphi\|_{B_{X_r}^{s_r, q_r}} = 0$. Indeed, since $q_r < \infty$, the Littlewood-Paley decomposition of φ implies that $\cap_{m=0}^{\infty} H_{X_r}^m(\mathbb{R}^N)$ is dense in $B_{X_r}^{s_r, q_r}(\mathbb{R}^N)$, where $H_{X_r}^m(\mathbb{R}^N) = (I - \Delta)^{-\frac{m}{2}} X_r(\mathbb{R}^N)$. Since $X_r(\mathbb{R}^N) \hookrightarrow B_{\infty\infty}^{-r}(\mathbb{R}^N)$, we have $H_{X_r}^m(\mathbb{R}^N) \hookrightarrow B_{X_r}^{m, \infty}(\mathbb{R}^N) \hookrightarrow B_{\infty\infty}^{m-r}(\mathbb{R}^N) \hookrightarrow L^{\infty}(\mathbb{R}^N)$ for $m \geq 1$, so that $H_{X_r}^{m+1}(\mathbb{R}^N) \hookrightarrow W^{m, \infty}(\mathbb{R}^N)$ for $m \geq 1$. Thus $C_{bu}(\mathbb{R}^N)$ is dense in $B_{X_r}^{s_r, q_r}(\mathbb{R}^N)$. Now for any $\varphi \in B_{X_r}^{s_r, q_r}(\mathbb{R}^N)$ and $\psi \in C_{bu}(\mathbb{R}^N) \subseteq L^{\infty}(\mathbb{R}^N) \subseteq X_r(\mathbb{R}^N)$, we write

$$\begin{aligned} \|e^{t\Delta}\varphi - \varphi\|_{B_{X_r}^{s_r, q_r}} &\leq \|e^{t\Delta}\varphi - e^{t\Delta}\psi\|_{B_{X_r}^{s_r, q_r}} + \|e^{t\Delta}\psi - \psi\|_{B_{X_r}^{s_r, q_r}} + \|\psi - \varphi\|_{B_{X_r}^{s_r, q_r}} \\ &\leq 2\|\varphi - \psi\|_{B_{X_r}^{s_r, q_r}} + \|e^{t\Delta}\psi - \psi\|_{X_r} \\ &\leq 2\|\varphi - \psi\|_{B_{X_r}^{s_r, q_r}} + \|e^{t\Delta}\psi - \psi\|_{\infty}. \end{aligned}$$

From this inequality and a similar argument as in the proof of Lemma 2.3, we see that the desired assertion follows. Finally, by the density of $L^\infty(\mathbb{R}^N)$ and $X_r(\mathbb{R}^N)$ in $B_{X_r}^{s_r, q_r}(\mathbb{R}^N)$ and a similar argument as in the proof of Lemma 2.3, we easily deduce that for any $\varphi \in B_{X_r}^{s_r, q_r}(\mathbb{R}^N)$,

$$\lim_{t \rightarrow 0^+} t^{\frac{1-r}{2}} \|e^{t\Delta} \varphi\|_{X_r} = 0 \quad \text{and} \quad \lim_{t \rightarrow 0^+} \sqrt{t} \|e^{t\Delta} \varphi\|_\infty = 0.$$

This proves the lemma. \square

The last three terms on the right-hand side of (3.2) will be treated with the same type of estimates. To establish such estimates, we need the following delicate inequality (see Lemma 20.1 of [18]):

Lemma 3.3 *Let $\theta \in (0, \frac{1}{2})$. Then for any $1 \leq q \leq \infty$, the operator $f \mapsto g$ defined by*

$$g(t) = \int_0^t \frac{1}{\sqrt{(t-\tau)\tau}} \left(\frac{t}{\tau}\right)^\theta f(\tau) d\tau$$

is bounded on $L^q((0, T), \frac{dt}{t})$. More precisely, there exists a constant $C > 0$ depending only on θ and q (independent of f and T) such that

$$\left(\int_0^T |g(t)|^q \frac{dt}{t} \right)^{\frac{1}{q}} \leq C \left(\int_0^T |f(t)|^q \frac{dt}{t} \right)^{\frac{1}{q}} \quad \text{if } 1 \leq q < \infty,$$

and $\|g\|_\infty \leq C \|f\|_\infty$ if $q = \infty$. \square

Lemma 3.4 *Assume that $\sqrt{t}\mathbf{v}(t) \in L^\infty([0, T], L^\infty(\mathbb{R}^N))$. Then we have the following assertions:*

(i) If $\mathbf{u} \in L^{q_r}([0, T], X_r(\mathbb{R}^N))$ then $B(\mathbf{u}, \mathbf{v}), B(\mathbf{v}, \mathbf{u}) \in L^{q_r}([0, T], X_r(\mathbb{R}^N)) \cap C([0, T], B_{X_r}^{s_r, q_r}(\mathbb{R}^N))$, and

$$\|B(\mathbf{u}, \mathbf{v})\|_{L_T^{q_r} X_r} + \|B(\mathbf{v}, \mathbf{u})\|_{L_T^{q_r} X_r} \lesssim \sup_{t \in [0, T]} \sqrt{t} \|\mathbf{v}(t)\|_\infty \|\mathbf{u}\|_{L_T^{q_r} X_r}, \quad (3.3)$$

$$\sup_{t \in [0, T]} \|B(\mathbf{u}, \mathbf{v})(t)\|_{B_{X_r}^{s_r, q_r}} + \sup_{t \in [0, T]} \|B(\mathbf{v}, \mathbf{u})(t)\|_{B_{X_r}^{s_r, q_r}} \lesssim \sup_{t \in [0, T]} \sqrt{t} \|\mathbf{v}(t)\|_\infty \|\mathbf{u}\|_{L_T^{q_r} X_r}. \quad (3.4)$$

(ii) If $t^{\frac{1-r}{2}} \mathbf{u}(t) \in L^\infty([0, T], X_r(\mathbb{R}^N))$ then $t^{\frac{1-r}{2}} B(\mathbf{u}, \mathbf{v}), t^{\frac{1-r}{2}} B(\mathbf{v}, \mathbf{u}) \in L^\infty([0, T], X_r(\mathbb{R}^N))$, $\sqrt{t} B(\mathbf{u}, \mathbf{v}), \sqrt{t} B(\mathbf{v}, \mathbf{u}) \in L^\infty([0, T], L^\infty(\mathbb{R}^N))$, and

$$\sup_{t \in [0, T]} t^{\frac{1-r}{2}} \|B(\mathbf{u}, \mathbf{v})\|_{X_r} + \sup_{t \in [0, T]} t^{\frac{1-r}{2}} \|B(\mathbf{v}, \mathbf{u})\|_{X_r} \lesssim \sup_{t \in [0, T]} \sqrt{t} \|\mathbf{v}(t)\|_\infty \sup_{t \in [0, T]} t^{\frac{1-r}{2}} \|\mathbf{u}(t)\|_{X_r}, \quad (3.5)$$

$$\sup_{t \in [0, T]} \sqrt{t} \|B(\mathbf{u}, \mathbf{v})(t)\|_\infty + \sup_{t \in [0, T]} \sqrt{t} \|B(\mathbf{v}, \mathbf{u})(t)\|_\infty \lesssim \sup_{t \in [0, T]} \sqrt{t} \|\mathbf{v}(t)\|_\infty \sup_{t \in [0, T]} t^{\frac{1-r}{2}} \|\mathbf{u}(t)\|_{X_r}. \quad (3.6)$$

Proof: The proofs of (3.3), (3.5) and (3.6) are contained in the proof of Theorem 20.2 of [18]. Here we only give the proof of (3.4) and the assertion that $B(\mathbf{u}, \mathbf{v}), B(\mathbf{v}, \mathbf{u}) \in C([0, T], B_{X_r}^{s_r, q_r}(\mathbb{R}^N))$.

Moreover, we only give the proof for $B(\mathbf{v}, \mathbf{u})$, because that for $B(\mathbf{u}, \mathbf{v})$ is similar. We first prove (3.4). We use the heat kernel characterization of $B_{X_r}^{s_r, q_r}(\mathbb{R}^N)$. First we have

$$\begin{aligned}
\|e^{\Delta} B(\mathbf{v}, \mathbf{u})(t)\|_{X_r} &\lesssim \int_0^t (1+t-\tau)^{-\frac{1}{2}} \|\mathbf{v}(\tau)\|_{\infty} \|\mathbf{u}(\tau)\|_{X_r} d\tau \\
&\lesssim \sup_{t \in [0, T]} \sqrt{t} \|\mathbf{v}(t)\|_{\infty} \int_0^t (1+t-\tau)^{-\frac{1}{2}} \tau^{-\frac{1}{2}} \|\mathbf{u}(\tau)\|_{X_r} d\tau \\
&\lesssim \sup_{t \in [0, T]} \sqrt{t} \|\mathbf{v}(t)\|_{\infty} \cdot \left(\int_0^{\infty} |1-\tau|^{-\frac{q_r'}{2}} \tau^{-\frac{q_r'}{2}} d\tau \right)^{\frac{1}{q_r}} \|\mathbf{u}\|_{L_T^{q_r} X_r} \\
&\lesssim \sup_{t \in [0, T]} \sqrt{t} \|\mathbf{v}(t)\|_{\infty} \|\mathbf{u}\|_{L_T^{q_r} X_r}, \quad \forall t \in [0, T].
\end{aligned}$$

Next, let \mathbf{u}^* be as before. Then for any $0 < s < 1$ and $0 \leq t \leq T$ we have

$$\begin{aligned}
\|e^{s\Delta} B(\mathbf{v}, \mathbf{u})(t)\|_{X_r} &\lesssim \int_0^t (t+s-\tau)^{-\frac{1}{2}} \|\mathbf{v}(\tau)\|_{\infty} \|\mathbf{u}(\tau)\|_{X_r} d\tau \\
&\lesssim \sup_{t \in [0, T]} \sqrt{t} \|\mathbf{v}(t)\|_{\infty} \int_0^{t+s} (t+s-\tau)^{-\frac{1}{2}} \tau^{-\frac{1}{2}} \|\mathbf{u}^*(\tau)\|_{X_r} d\tau,
\end{aligned}$$

so that

$$s^{\frac{1-r}{2}} \|e^{s\Delta} B(\mathbf{v}, \mathbf{u})(t)\|_{X_r} \lesssim \sup_{t \in [0, T]} \sqrt{t} \|\mathbf{v}(t)\|_{\infty} \int_0^{t+s} (t+s-\tau)^{-\frac{1}{2}} \tau^{-\frac{1}{2}} \left(\frac{t+s}{\tau} \right)^{\frac{1-r}{2}} \cdot \tau^{\frac{1-r}{2}} \|\mathbf{u}^*(\tau)\|_{X_r} d\tau.$$

Using Lemma 3.3, we get

$$\begin{aligned}
\left[\int_0^1 \|e^{s\Delta} B(\mathbf{v}, \mathbf{u})(t)\|_{X_r}^{q_r} ds \right]^{\frac{1}{q_r}} &= \left[\int_0^1 \left(s^{\frac{1-r}{2}} \|e^{s\Delta} B(\mathbf{v}, \mathbf{u})(t)\|_{X_r} \right)^{q_r} \frac{ds}{s} \right]^{\frac{1}{q_r}} \\
&\lesssim \sup_{t \in [0, T]} \sqrt{t} \|\mathbf{v}(t)\|_{\infty} \left[\int_0^{\infty} \left(\tau^{\frac{1-r}{2}} \|\mathbf{u}^*(\tau)\|_{X_r} \right)^{q_r} \frac{d\tau}{\tau} \right]^{\frac{1}{q_r}} \\
&\lesssim \sup_{t \in [0, T]} \sqrt{t} \|\mathbf{v}(t)\|_{\infty} \|\mathbf{u}\|_{L_T^{q_r} X_r}.
\end{aligned}$$

This proves (3.4).

Having proved (3.4), the assertion that $B(\mathbf{v}, \mathbf{u}) \in C([0, T], B_{X_r}^{s_r, q_r}(\mathbb{R}^N))$ follows from a similar argument as in the proof of the corresponding assertion in Lemma 2.6. \square

We are now ready to give the proof of Theorem 3.1.

First, by Lemma 3.4 we see that for a given function \mathbf{a} on $\mathbb{R}^N \times [0, T]$ such that $\sqrt{t}\mathbf{a}(t) \in L^{\infty}([0, T], L^{\infty}(\mathbb{R}^N))$, the map

$$L(\mathbf{a}) : \mathbf{u} \mapsto -B(\mathbf{a}, \mathbf{u}) - B(\mathbf{u}, \mathbf{a})$$

is a bounded linear operator in \mathcal{V}_T , with norm bounded by $C \sup_{t \in [0, T]} \sqrt{t} \|\mathbf{a}(t)\|_{\infty}$, where C is a positive constant independent of \mathbf{a} and T . It follows that if

$$C \sup_{t \in [0, T]} \sqrt{t} \|\mathbf{a}(t)\|_{\infty} \leq \frac{1}{2}, \quad (3.7)$$

then $I - L(\mathbf{a})$ is invertible, and $\|[I - L(\mathbf{a})]^{-1}\|_{\mathcal{L}(\mathcal{V}_T)} \leq 2$. Clearly, the condition (3.7) is satisfied in each of the following two situations:

- Given \mathbf{a} , restrict our discussion in a possibly smaller interval $[0, T_1]$ ($0 < T_1 \leq T$) so that $C \sup_{t \in [0, T_1]} \sqrt{t} \|\mathbf{a}(t)\|_\infty \leq \frac{1}{2}$. Existence of a such T_1 is guaranteed by the condition $\lim_{t \rightarrow 0^+} \sqrt{t} \|\mathbf{a}(t)\|_\infty = 0$. In this case we need to replace all T with T_1 in the succeeding discussion.
- Given $T > 0$, restrict our discussion to those \mathbf{a} such that $C \sup_{t \in [0, T_1]} \sqrt{t} \|\mathbf{a}(t)\|_\infty \leq \frac{1}{2}$. This is a smallness assumption on \mathbf{a} .

In both situations, it can be easily seen that when considering solutions in \mathcal{V}_T , the equation (3.2) is equivalent to the following equation:

$$\mathbf{u}(t) = [I - L(\mathbf{a})]^{-1} [e^{t\Delta} \mathbf{u}_0 - B(\mathbf{u}, \mathbf{u})]. \quad (3.8)$$

Letting $\mathcal{G}(\mathbf{u})$ be the right-hand side of (3.8), by using Lemmas 3.2 and 3.4 we see that \mathcal{G} maps \mathcal{V}_T to itself, and the following estimates hold:

$$\begin{aligned} \|\mathcal{G}(\mathbf{u})\|_{\mathcal{V}_T} &\leq 2\|e^{t\Delta} \mathbf{u}_0\|_{\mathcal{V}_T} + C\|\mathbf{u}\|_{\mathcal{V}_T}^2, \\ \|\mathcal{G}(\mathbf{u}) - \mathcal{G}(\mathbf{v})\|_{\mathcal{V}_T} &\leq C[\|\mathbf{u}\|_{\mathcal{V}_T} + \|\mathbf{v}\|_{\mathcal{V}_T}]\|\mathbf{u} - \mathbf{v}\|_{\mathcal{V}_T}. \end{aligned}$$

Thus, a similar argument as in the proof of Theorem 2.1 shows that \mathcal{G} has a fixed point in a small closed neighborhood of the origin of \mathcal{V}_T in each of the following two situations:

- Given $\mathbf{u}_0 \in B_{X_r}^{s_r, q_r}(\mathbb{R}^N)$, restrict our discussion in a possibly smaller interval $[0, T_2]$ ($0 < T_2 \leq T_1$) so that $2\|e^{t\Delta} \mathbf{u}_0\|_{\mathcal{V}_{T_2}} \leq \varepsilon$, where ε is an arbitrarily chosen positive number such that $4C\varepsilon < 1$. Existence of a such number T_2 is ensured by Lemma 3.2. In this case we need to replace all T with T_2 in the succeeding discussion.
- Given $T > 0$, restrict our discussion to those $\mathbf{u}_0 \in B_{X_r}^{s_r, q_r}(\mathbb{R}^N)$ such that $2C_0\|\mathbf{u}_0\|_{B_{X_r}^{s_r, q_r}} \leq \varepsilon$, where C_0 is the constant that appears in the inequality $\|e^{t\Delta} \mathbf{u}_0\|_{\mathcal{V}_T} \leq C_0\|\mathbf{u}_0\|_{B_{X_r}^{s_r, q_r}}$, and ε is as above. This is a smallness assumption on \mathbf{u}_0 .

Consequently, we obtain a solution of (3.2) in \mathcal{V}_T . Since by Lemmas 3.2 and 3.4, for any $\mathbf{u}_0 \in B_{X_r}^{s_r, q_r}(\mathbb{R}^N)$ and $\mathbf{u} \in \mathcal{V}_T$, all terms on the right-hand side of (3.2) belong to \mathcal{W}_T , we actually have obtained a solution of (3.2) in \mathcal{W}_T . Moreover, since $e^{t\Delta}$ is also a C_0 -semigroup in $B_{\tilde{X}_r}^{s_r, q_r}(\mathbb{R}^N)$, a similar discussion shows that if $\mathbf{u}_0 \in B_{\tilde{X}_r}^{s_r, q_r}(\mathbb{R}^N)$ then the solution \mathbf{u} belongs to $C([0, T], B_{\tilde{X}_r}^{s_r, q_r}(\mathbb{R}^N)) \cap \mathcal{V}_T$. This completes the proof of Theorem 3.1. \square

4 The proof of Theorem 1.4

In this section we give the proof of Theorem 1.4. We first prove the following result:

Theorem 4.1 *Let $0 < T < \infty$ and $\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2$ be given, where $\mathbf{a}_1 \in L^2([0, T], L^\infty(\mathbb{R}^N))$ and $\mathbf{a}_2 \in L^{q_r}([0, T], X_r(\mathbb{R}^N))$ ($0 < r < 1$, $q_r = \frac{2}{1-r}$). Then for any $\mathbf{u}_0 \in L^2(\mathbb{R}^N)$ satisfying the condition $\nabla \cdot \mathbf{u}_0 = 0$, the problem (3.1) has a weak solution $\mathbf{u} \in C_w([0, T], L^2(\mathbb{R}^N)) \cap L^2([0, T], H^1(\mathbb{R}^N))$ satisfying the following generalized energy inequality:*

$$\sup_{0 \leq t \leq T} \|\mathbf{u}(t)\|_2^2 + \int_0^T \|\nabla \mathbf{u}(t)\|_2^2 dt \leq \|\mathbf{u}_0\|_2^2 \exp \left[C \int_0^t \left(\|\mathbf{a}_1(\tau)\|_\infty^2 + \|\mathbf{a}_2(\tau)\|_{X_r}^{\frac{2}{1-r}} \right) d\tau \right]. \quad (4.1)$$

Proof. Choose a sequence of divergence-free vector functions $\{\mathbf{w}_k\}_{k=1}^\infty \subseteq \cap_{m=0}^\infty H^m(\mathbb{R}^N)$ such that they form a normalized orthogonal basis of $\mathbb{P}L^2(\mathbb{R}^N)$. Let $\mathbf{u}_0 = \sum_{k=0}^\infty c_k \mathbf{w}_k$ be the Fourier expansion of \mathbf{u}_0 with respect to this basis of $\mathbb{P}L^2(\mathbb{R}^N)$. For each n we define an approximate solution \mathbf{u}_n of the problem (3.1) on the time interval $[0, T]$ as follows:

$$\mathbf{u}_n(x, t) = \sum_{k=1}^n g_{kn}(t) \mathbf{w}_k(x), \quad x \in \mathbb{R}^N, \quad t \in [0, T],$$

where $\{g_{kn}\}_{k=1}^n$ is the solution of the initial value problem

$$\begin{cases} g'_{kn}(t) = \sum_{j=1}^n b_{jk}(t) g_{jn}(t) + \sum_{i,j=1}^n c_{ijk} g_{in}(t) g_{jn}(t), & 0 < t < T, \quad k = 1, 2, \dots, n, \\ g_{kn}(0) = c_k, & k = 1, 2, \dots, n, \end{cases} \quad (4.2)$$

where

$$\begin{cases} b_{jk}(t) = - \int_{\mathbb{R}^N} \nabla \mathbf{w}_j(x) \cdot \nabla \mathbf{w}_k(x) dx + 2 \int_{\mathbb{R}^N} [\mathbf{a}(x, t) \otimes_s \mathbf{w}_j(x)] \cdot [\nabla \otimes \mathbf{w}_k(x)] dx, \\ c_{ijk} = \int_{\mathbb{R}^N} [\mathbf{w}_i(x) \otimes \mathbf{w}_j(x)] \cdot [\nabla \otimes \mathbf{w}_k(x)] dx \end{cases}$$

($i, j, k = 1, 2, \dots, n$). Note that by the assumption on \mathbf{a} we have $b_{jk} \in L^2(0, T)$ ($i, j, k = 1, 2, \dots, n$), so that the problem (4.2) has a unique solution $\{g_{kn}\}_{k=1}^n$ defined in a maximal interval I which is either $[0, T]$ or $[0, T^*)$ with $0 < T^* \leq T$, such that $g_{kn} \in C(I)$ and $g'_{kn} \in L^2_{loc}(I)$, $k = 1, 2, \dots, n$, and in the second case $g_{kn}(t)$ blows up as $t \rightarrow T^{*-}$. By making a priori estimates we can ensure that the second case cannot occur. Indeed, (4.2) can be rewritten as follows (note that $\mathbb{P}\Delta \mathbf{u}_n = \Delta \mathbb{P} \mathbf{u}_n = \Delta \mathbf{u}_n$):

$$\begin{cases} \partial_t \mathbf{u}_n - \Delta \mathbf{u}_n + \mathbb{P}_n [2\nabla \cdot (\mathbf{a} \otimes_s \mathbf{u}_n) + \nabla \cdot (\mathbf{u}_n \otimes \mathbf{u}_n)] = 0, & x \in \mathbb{R}^N, \quad t > 0, \\ \mathbf{u}_n(x, 0) = \mathbf{u}_{0n}(x), & x \in \mathbb{R}^N, \end{cases} \quad (4.3)$$

where \mathbb{P}_n denotes the projection from $L^2(\mathbb{R}^N)$ to its n -dimensional subspace spanned by $\{\mathbf{w}_k\}_{k=1}^n$, and $\mathbf{u}_{0n} = \sum_{k=1}^n c_k \mathbf{w}_k$. It is easy to see that

$$\begin{aligned} \int_{\mathbb{R}^N} [\mathbf{a}_1(x, t) \otimes_s \mathbf{u}_n(x, t)] \cdot [\nabla \otimes \mathbf{u}_n(x, t)] dx &\leq \|\mathbf{a}_1(t)\|_\infty \|\mathbf{u}_n(t)\|_2 \|\nabla \mathbf{u}_n(t)\|_2, \\ \int_{\mathbb{R}^N} [\mathbf{a}_2(x, t) \otimes_s \mathbf{u}_n(x, t)] \cdot [\nabla \otimes \mathbf{u}_n(x, t)] dx &\leq \|\mathbf{a}_2(t)\|_{\dot{X}_r} \|\mathbf{u}_n(t)\|_2^{1-r} \|\nabla \mathbf{u}_n(t)\|_2^{1+r}. \end{aligned}$$

Using these inequalities and a standard argument we can easily prove that

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{u}_n(t)\|_2^2 \leq -\frac{1}{2} \|\nabla \mathbf{u}_n(t)\|_2^2 + C \left(\|\mathbf{a}_1(t)\|_\infty^2 + \|\mathbf{a}_2(t)\|_{\dot{X}_r}^{\frac{2}{1-r}} \right) \|\mathbf{u}_n(t)\|_2^2.$$

Hence

$$\|\mathbf{u}_n(t)\|_2^2 + \int_0^t \|\nabla \mathbf{u}_n(\tau)\|_2^2 d\tau \leq \|\mathbf{u}_{0n}\|_2^2 \exp \left[C \int_0^t \left(\|\mathbf{a}_1(\tau)\|_\infty^2 + \|\mathbf{a}_2(\tau)\|_{\dot{X}_r}^{\frac{2}{1-r}} \right) d\tau \right]. \quad (4.4)$$

From this estimate, the desired assertion follows immediately.

By (4.4), we easily deduce that for any $2 \leq p \leq \frac{2N}{N-2}$ and $2 \leq q \leq \infty$ satisfying the relation $\frac{N}{p} + \frac{2}{q} = \frac{N}{2}$, $\{\mathbf{u}_n\}_{n=1}^\infty$ is bounded in $L^q([0, T], L^p(\mathbb{R}^N)) \cap L^2([0, T], H^1(\mathbb{R}^N))$. This implies that $\{\nabla \cdot (\mathbf{a} \otimes_s \mathbf{u}_n)\}_{n=1}^\infty$, $\{\nabla \cdot (\mathbf{u}_n \otimes \mathbf{u}_n)\}_{n=1}^\infty$ and $\{\Delta \mathbf{u}_n\}_{n=1}^\infty$ are bounded in $L^2([0, T], H^{-2}(\mathbb{R}^N))$. Using the equation (4.3), we further deduce that $\{\partial_t \mathbf{u}_n\}_{n=1}^\infty$ is bounded in $L^2([0, T], H^{-2}(\mathbb{R}^N))$. From these assertions and Theorem 2.1 in Chapter III of [21], we conclude that there is a subsequence $\{\mathbf{u}_{n_k}\}_{k=1}^\infty$ of $\{\mathbf{u}_n\}_{n=1}^\infty$, and a divergence-free vector function \mathbf{u} such that

- $\mathbf{u}_{n_k} \rightarrow \mathbf{u}$ *-weakly in $L^\infty([0, T], L^2(\mathbb{R}^N))$ and weakly in $L^2([0, T], H^1(\mathbb{R}^N))$;
- $\partial_t \mathbf{u}_{n_k} \rightarrow \partial_t \mathbf{u}$ weakly in $L^2([0, T], H^{-2}(\mathbb{R}^N))$;
- For any bounded measurable set $Q \subseteq \mathbb{R}^N$ and any $2 \leq p < \frac{2N}{N-2}$, $\mathbf{u}_{n_k} \rightarrow \mathbf{u}$ strongly in $L^2([0, T], L^p(Q))$.

These assertions imply that $\mathbf{u} \in L^\infty([0, T], L^2(\mathbb{R}^N)) \cap L^2([0, T], H^1(\mathbb{R}^N))$ and $\partial_t \mathbf{u} \in L^2([0, T], H^{-2}(\mathbb{R}^N))$, which further implies that $\mathbf{u} \in C([0, T], H^{-1/2}(\mathbb{R}^N))$, so that $\mathbf{u} \in C_w([0, T], L^2(\mathbb{R}^N))$. Moreover, by using the last assertion listed above, the boundedness of $\{\mathbf{u}_n\}_{n=1}^\infty$ in $L^{\frac{2(N+2)}{N}}(\mathbb{R}^N \times [0, T])$, and the Vitali convergence theorem (cf. Corollary A.2 of [6]), we conclude that for any $1 \leq p < 2(N+2)/N$ and any bounded measurable set $Q \subseteq \mathbb{R}^N$, $\mathbf{u}_{n_k} \rightarrow \mathbf{u}$ strongly in $L^p(Q \times [0, T])$. This immediately implies that for any $1 \leq p < (N+2)/N$ and any bounded measurable set $Q \subseteq \mathbb{R}^N$, $\mathbf{u}_{n_k} \otimes \mathbf{u}_{n_k} \rightarrow \mathbf{u} \otimes \mathbf{u}$ strongly in $L^p(Q \times [0, T])$. It follows that

$$\mathbf{u}_{n_k} \otimes \mathbf{u}_{n_k} \rightarrow \mathbf{u} \otimes \mathbf{u} \text{ weakly in distribution sense,}$$

so that also $\mathbf{u}_{n_k} \otimes \mathbf{u}_{n_k} \rightarrow \mathbf{u} \otimes \mathbf{u}$ weakly in $L^{1+\frac{2}{N}}(\mathbb{R}^N \times [0, T])$, because $\mathbf{u}_{n_k} \otimes \mathbf{u}_{n_k}$ is bounded in $L^{1+\frac{2}{N}}(\mathbb{R}^N \times [0, T])$. Having obtained this assertion, we can now let $n = n_k \rightarrow \infty$ in (4.3) to conclude that \mathbf{u} is a weak solution of the problem (3.1) in $\mathbb{R}^N \times [0, T]$. The inequality (4.1) is an immediate consequence of (4.4) and the Fatou lemma. The proof of Theorem 3.1 is complete. \square

Proof of Theorem 1.4: Let $0 < T < \infty$ be given. We first assume that $\mathbf{u}_0 = \mathbf{v}_0 + \mathbf{w}_0 + \mathbf{z}_0$, $\mathbf{v}_0 \in B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)$, $\mathbf{w}_0 \in B_{\dot{X}_r}^{s_r, q_r}(\mathbb{R}^N)$, $\mathbf{z}_0 \in L^2(\mathbb{R}^N)$, $\nabla \cdot \mathbf{v}_0 = \nabla \cdot \mathbf{w}_0 = \nabla \cdot \mathbf{z}_0 = 0$, $\|\mathbf{v}_0\|_{B_{\infty\infty}^{-1(\ln)}} < \varepsilon_1$ and $\|\mathbf{w}_0\|_{B_{\dot{X}_r}^{s_r, q_r}} < \varepsilon_2$, where ε_1 and ε_2 are positive numbers to be specified later. Consider the problem (1.1) with \mathbf{u}_0 replaced by \mathbf{v}_0 . By Theorem 2.1, we see that if ε_1

is sufficiently small then that problem has a unique solution which we denote as \mathbf{v} , such that $\mathbf{v} \in C([0, T], B_{\infty\infty}^{-1(\ln)}(\mathbb{R}^N)) \cap \mathcal{X}_T$. Note that $\mathbf{v} \in \mathcal{X}_T$ implies that $\mathbf{v} \in L^2([0, T], L^\infty(\mathbb{R}^N))$, $\sqrt{t}\mathbf{v} \in L^\infty(\mathbb{R}^N \times [0, T])$ and $\lim_{t \rightarrow 0^+} \sqrt{t}\|\mathbf{v}(t)\|_\infty = 0$. Moreover, from the proof of Theorem 2.1 we see that $\sup_{0 \leq t \leq T} \|\mathbf{v}(t)\|_{B_{\infty\infty}^{-1(\ln)}} + \sup_{0 \leq t \leq T} \sqrt{t}\|\mathbf{v}(t)\|_\infty + \|\mathbf{v}\|_{L_T^2 L_x^\infty} \leq C\varepsilon_1$, where C is a constant depending only on the dimension N . Next consider the problem (3.1) with \mathbf{a} and \mathbf{u}_0 replaced by \mathbf{v} and \mathbf{w}_0 , respectively. By Theorem 3.1 (ii), we see that if ε_1 and ε_2 are sufficiently small then that problem has a unique solution which we denote as \mathbf{w} , such that $\mathbf{w} \in C([0, T], B_{\tilde{X}_r}^{s_r, q_r}(\mathbb{R}^N)) \cap L^{q_r}([0, T], X_r(\mathbb{R}^N))$ (we neglect the other properties of \mathbf{w} because we do not use them below). We now consider the problem (3.1) again, but replace \mathbf{a} and \mathbf{u}_0 with $\mathbf{v} + \mathbf{w}$ and \mathbf{z}_0 , respectively. By using Theorem 4.1 with $\mathbf{a}_1 = \mathbf{v}$ and $\mathbf{a}_2 = \mathbf{w}$, we see that that problem has a weak solution, which we denote as \mathbf{z} , such that $\mathbf{z} \in C_w([0, T], L^2(\mathbb{R}^N)) \cap L^2([0, T], H^1(\mathbb{R}^N))$. Now let $\mathbf{u} = \mathbf{v} + \mathbf{w} + \mathbf{z}$. Then as one can easily verify, \mathbf{u} is the desired solution of the original problem (1.1). This proves the first assertion of Theorem 1.4.

The second assertion is an easy consequence of the first one. Indeed, since $C_0^\infty(\mathbb{R}^N)$ is dense in $C_0^{-1(\ln)}(\mathbb{R}^N)$, by using Proposition 12.1 of [18] we can split \mathbf{v}_0 into a sum $\mathbf{v}_0 = \mathbf{v}'_0 + \mathbf{v}''_0$, such that $\mathbf{v}'_0 \in C_0^{-1(\ln)}(\mathbb{R}^N)$, $\mathbf{v}''_0 \in L^2(\mathbb{R}^N)$, $\nabla \cdot \mathbf{v}'_0 = \nabla \cdot \mathbf{v}''_0 = 0$, and $\|\mathbf{v}'_0\|_{B_{\infty\infty}^{-1(\ln)}}$ is as small as we want. Similarly, since $C_0^\infty(\mathbb{R}^N)$ is dense in $B_{\tilde{X}_r}^{s_r, q_r}(\mathbb{R}^N)$, we can split \mathbf{w}_0 into a sum $\mathbf{w}_0 = \mathbf{w}'_0 + \mathbf{w}''_0$, such that $\mathbf{w}'_0 \in B_{\tilde{X}_r}^{s_r, q_r}(\mathbb{R}^N)$, $\mathbf{w}''_0 \in L^2(\mathbb{R}^N)$, $\nabla \cdot \mathbf{w}'_0 = \nabla \cdot \mathbf{w}''_0 = 0$, and $\|\mathbf{w}'_0\|_{B_{\tilde{X}_r}^{s_r, q_r}}$ is as small as we want. Thus, by using the first assertion we immediately obtain the second assertion. This completes the proof of Theorem 1.4. \square

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